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FORMATION AND DECAY OF SHOCK WAVES

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INTRODUCTION

The present preliminary report, by K. O. Friedrichs, was prepared under Navy Contract N6ori-201, Task Order No. 1. It contains an approximate treatment of interactions of continuous motion in a fluid with shock waves of moderate strength. In particular, the general method developed is applied to the problem of formation of shocks in a compression wave, to the decay of shocks under the influence of rarefaction waves, and to the propagation of N-waves and other flow patterns as they occur in supersonic flow past airfoils.

Apart from its theoretical interest the method leads to simple procedures for numerical calculation.

In the appendices of a more detailed publication the mathematical theory will be amplified, further applications, especially to the theory of water waves in shallow channels will be presented, and the accuracy of the approximate results will be demonstrated.

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FORMATION AND DECAY OF SHOCK WAVES

By

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To determine the interaction of shocks with continuous gas flow is a difficult problem which, however, can be handled, with great labor, by the method of finite differences. If the shock is weak or of medium strength the treatment can be greatly simplified in good approximation, as Chandrasekhar has observed.(*) He obtained an explicit approximate solution of the interaction problem in a special case. It will be shown in the present paper that a slightly modified approach enables one to obtain, quite generally, explicit analytic solutions of such interaction problems in good approximation. This approach opens up the possibility of handling analytically a number of interaction problems for one-dimensional unsteady and two-dimensional steady gas flow, which so far have been inaccessible.

1. One such problem is that of the formation of a shock wave. Suppose a cylindrical tube full of gas is closed at one end by a piston which is pushed into the gas with finite acceleration. We know that a simple condensation wave is propagated into the gas and that at some later time this wave leads to a discontinuity which is resolved by the development of a shock. The strength of this shock begins with the value zero and builds up gradually. Therefore any method good for

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(*) S. Chandrasekhar, On the decay of plane shock waves. Aberdeen Proving Ground, Ballistics Research Laboratory, Report No. 423, 8 November 1943.

treating weak or medium shocks should be applicable at early and intermediate stages of the process. Let us describe the motion in an (x,t)-diagram and assume the acceleration to be constant up to a certain time and then zero again. The piston is then represented by a section of a parabola $x = \frac{1}{2} bt^2$, continued as a tangent from some point on. The resulting flow, a "simple" compression wave, may be described with the aid of the characteristic lines issuing from the piston curve. They are straight lines and carry constant values of all pertinent quantities. These straight lines, continued indefinitely, form an envelope with a cusp at some point on the first characteristic $x = c_0 t$ issuing from the point $x = 0$, $t = 0$ (cf. Manual(*), III 25, p. 60). The cusp region between the two branches of the envelope, of which one is a section of the first characteristic $x = c_0 t$, is covered three times by the straight lines. The continuation of the flow from the left or from the right of the cusp would lead to a different description. This contradiction is resolved by the appearance of a shock across which the pertinent quantities of the flow undergo a jump. The shock influences the flow behind it; more specifically, the flow differs from the simple compression wave described in an increasing zone represented by the region in the (x,t)-plane obtained by drawing the "cross-characteristic" backward from the cusp to the piston curve. (This in-

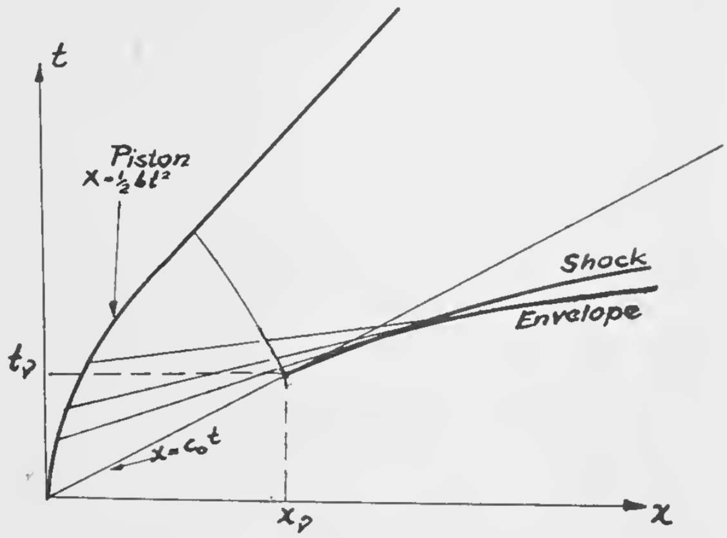

Fig. 1

Formation of shock in wave created by piston moving with constant acceleration.

(*) Supersonic Flow and Shock Waves, For the Applied Mathematics Panel, National Research Defense Committee, by the Applied Mathematics Group, New York University, 1944.

fluence of the shock on the flow can to a certain extent be described in terms of a "reflected wave".) The formation of this shock can now easily be described on the basis of the method to be presented.(*)

2. The problem of the <u>decay of a shock wave</u> is encountered when the piston, through whose motion a shock was formed in the tube, is arrested or retracted. The rarefaction wave sent out during the process of deceleration follows the shock front with sound speed; hence it eventually catches up with it, and diminishes its strength. Let us assume that the piston was instantaneously accelerated to constant velocity; then a constant shock results. Arresting the piston produces a simple rarefaction wave. (The total phenomenon: shock front followed by a rarefaction wave, which is modified by the shock, is often called "shock wave".) The width of an undisturbed simple rarefaction wave at any time increases linearly with time. The shock front, through its diminishing velocity, reduces the rate of increase of the width of the wave zone. Whether or not the width of this "decaying" shock wave zone actually decreases and whether or not the interaction process comes to an end at a finite time has often been asked. The approximate treatment to be presented gives a definite (though not rigorously proved) answer: <u>the width of the</u>

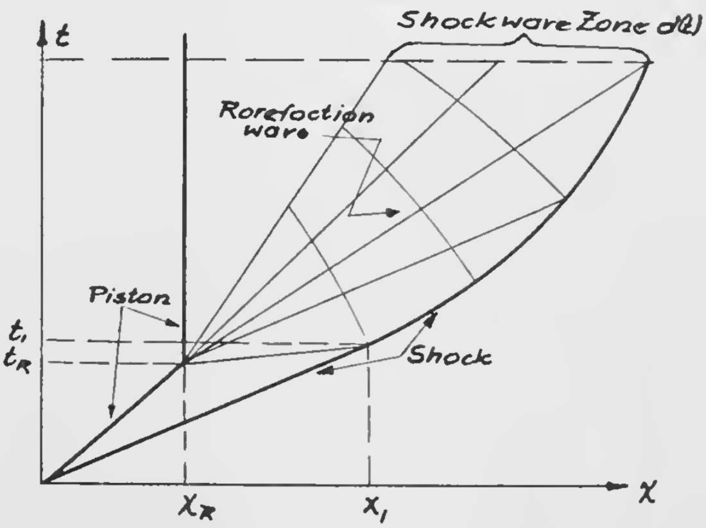

Fig. 2

Decay of shock wave due to arrest of piston.

(*) v. Neumann indicated in one of his reports some years ago that Calkin made a study of the formation problem, but no report about this study is available.

decaying shock wave zone increases like the square root, and the strength of the shock decreases like the reciprocal of the square root, of the time elapsed since the start of the process.

Other problems of formation and decay of shocks, such as the problem of "N-waves" (Sec. 7) and the problem of the decay of a two-dimensional steady shock (Sec. 10), will be described later on.

3. At first we shall present the method of treatment to be employed. The method is based on the fact that the shock transition relation agrees in second order with the simple wave transition relations. More specifically: consider forward facing shocks of varying strength with the same state ahead of them and expand all pertinent quantities behind the wave in powers of any quantity that can serve as a measure of shock strength, such as the velocity difference. Do the same with the quantities behind a forward facing simple wave having the same state ahead of it as the shock. The expansions of sound speed c and entropy s with respect to powers of the strength then agree for the shock and the simple wave up to terms of second order; the same is true for the expansion of pressure and density. Since the entropy is constant across a simple wave, this statement implies the well known fact that the change of entropy across a shock is of third order. Since the Riemann invariant (see below), is constant across a simple wave, the change of the Riemann invariant across a shock is also of third order. This statement deserves considerable emphasis, which it has not received in the literature where it was occasionally, but somewhat casually, mentioned. (For its derivation see Appendix BI.)

The method to be presented consists of two steps. In the first step we replace two shock transition relations by the corresponding transition relations for simple waves. Specifically, we require that the entropy and the appropriate Riemann invariant do not change across the shock. Owing to

this simplification there is no back reaction of the shock on the flow behind it, as we shall justify in detail below. Consequently, we may describe the flow in the following manner. We first determine the simple wave independently of the shock and then draw the shock line in the wave region in an appropriate manner.

Suppose at the time $t = 0$, the velocity u and sound speed c are given as functions of x so as to satisfy the relations

$$(1) \qquad \mu^2 u(x) - (1 - \mu^2)\, c(x) = \text{const.}$$

for a forward facing simple wave (cf. Manual III-20). Here and in the following we use the abbreviation $\mu^2 = \frac{\gamma-1}{\gamma+1}$. The left member of equation (1) is just the Riemann invariant mentioned above. At the point $x = 0$ the quantities u and c shall suffer a discontinuity. We characterize by the subscripts (o) and (1) the states at the positive and negative side of the cross section $x = 0$. The requirement that the discontinuity involves a compression and that the gas crosses this cross section from the positive to the negative side is then expressed through the conditions

$$u_o < u_1 , \quad c_o < c_1 .^{(*)}$$

The discontinuity is therefore a forward facing shock; the shock relations are satisfied up to the second order in the shock strength.

We now determine a simple wave flow for $t > 0$ by passing through each point $x = \xi$ on the line $t = 0$ the forward characteristic with the slope $\frac{dx}{dt} = u(\xi) + c(\xi)$, and let u and c be

(*) We do not admit here the equality sign; the subsequent results remain valid, however, also for a point of continuity if the derivatives $\frac{du}{dx}$ and $\frac{dc}{dx}$ become infinite in an appropriate manner.

constant along it. We set accordingly

$$(2) \qquad x = \{u(\xi) + c(\xi)\} t + \xi , \ u = u(\xi), \ c = c(\xi) .$$

Owing to the compressive character of the discontinuity we have $u_o + c_o < u_1 + c_1$, and hence the two wave regions in the (x,t)-plane obtained from the two parts $x > 0$ and $x < 0$ overlap. The shock line is now to be drawn in the common region in an appropriate way such that the mutilated wave regions fit together.

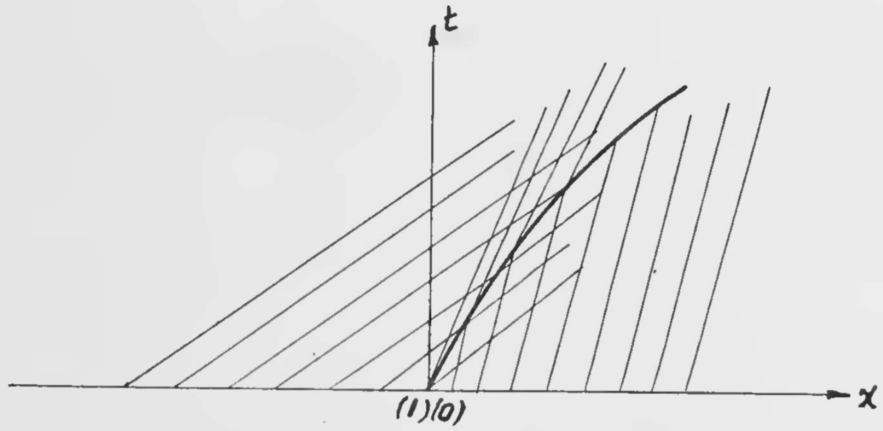

Fig. 3

Two forward facing simple waves separated by shock.

4. How to determine the path of the shock is best seen in case the state of the gas ahead of the shock front is constant: $u = u_o = 0$, $c = c_o$ for $x > 0$, initially, and hence also later on. Relation (1) then entails

$$(3) \qquad u = (1 - \mu^2)(u + c - c_o), \ c = c_o + \mu^2(u + c - c_o).$$

The relationship between shock velocity U and gas velocity u behind the shock is exactly given by

$$(4) \qquad u = \left(1 - \frac{2}{\mu}\right)(U - c_o^2/U) ,$$

see Manual III, Art.36(46), or using (3), by

$$(5) \qquad \frac{u+c}{c_o} - 1 = \frac{U}{c_o} - \frac{c_o}{U} .$$

Since the velocity behind the shock front is a given function $u(\xi)$ of the initial parameter ξ we can determine U from (3) or (5) as a function $U(\xi)$ of ξ. Thus we have for the shock

$$(6) \qquad \frac{dx}{dt} = U(\xi) .$$

We now consider the quantity t along the shock line as function of ξ and express x as function of ξ by (2). Differentiation of equation (2) with respect to ξ yields

$$U \frac{dt}{d\xi} = \left\{u + c\right\} \frac{dt}{d\xi} + \left\{u_\xi + c_\xi\right\} t + 1$$

or

$$(7) \qquad \left\{U - u - c\right\} \frac{dt}{d\xi} - \left\{u_\xi + c_\xi\right\} t = 1 .$$

<u>This is an ordinary linear differential equation,</u> the "shock differential equation", for t as function of ξ, which yields a unique solution by virtue of the initial condition $t = 0$ for $\xi = 0$. Thus the propagation of the shock is determined.

Introducing the function

$$(8) \qquad \zeta = \zeta(\xi) = \int^{\xi} \frac{d(u+c)}{c+u-U}$$

with any convenient constant of integration, the solution can be written in the form

$$(9) \qquad t = e^{-\zeta(\xi)} \int_{\xi}^{0} e^{\zeta(\xi)} \frac{d\xi}{c+u-U} .$$

Insertion of this relation into (2) yields x as function of ξ. Thus the path of the shock front is given in parametric representation.

For the numerical treatment various simplifications are useful to evaluate the integrals. Chandrasekhar, who has developed the method in connection with a special case, uses the quantity U instead of ξ as parameter. We found it more convenient to employ

$$(10) \qquad \sigma = \frac{u+c-c_o}{c_o}$$

as parameter and to set

$$(11) \qquad U = c_o \left\{1 + \frac{1}{2}\sigma + \frac{1}{8}\sigma^2\right\} ,$$

a formula which agrees with (5) up to terms of third order as is easily verified (cf. Appendix AI). (Note that the dependence of U/c_o on $(u + c)/c_o$ through (5) or (11) and the dependence of ζ on U/c_o is independent of γ.) Inserting (11) into (8) we find

$$(12) \qquad \zeta = 2 \log \frac{\sigma}{4-\sigma} , \quad e^{-\zeta} = \left(\frac{4-\sigma}{\sigma}\right)^2$$

whence by (9),

$$(13) \qquad c_o t = 8\left(\frac{4-\sigma}{\sigma}\right)^2 \int_{\xi}^{0} \frac{\sigma \, d\xi}{(4-\sigma)^3} .$$

(See Appendix AI.)

5. In the two problems of formation and decay mentioned in the introduction, we just have the situation that the state ahead of the shock front is constant. Hence these two problems can be treated by the method outlined. The results for these two problems are derived in Appendices AII and AIII. We give here a brief account.

In the <u>wave produced by a piston moving with constant</u>

<u>acceleration</u>, $x = \frac{1}{2} bt^2$, a shock develops at the time $t_\vartheta = (1 - \mu^2) \frac{c_o}{a}$, at the position $x_\vartheta = (1 - \mu^2) \frac{c_o^2}{a}$, (see Fig. 1). The path of the shock is (cf. Appendix AII), given by

$$(14) \quad x - x_\vartheta = c_o \left\{ (t - t_\vartheta) + \frac{3}{8(1-\mu^4)} \frac{b(t-t_\vartheta)^2}{c_o} + \right.$$

$$\left. + \frac{9}{128(1-\mu^4)^2} \frac{b^2(t-t_\vartheta)^3}{c_o^2} \right\}.$$

One observes that the acceleration of the shock front is initially about 3/4 of the value b of the acceleration of the piston, but then increases.

6. For the <u>decaying shock wave</u> (cf. Appendix AIII) we obtain more striking results. Suppose the piston is first moved with a velocity u_1 and stopped at a time t_R at $x = x_R = u_1 t_R$, (see Fig. 2). A shock starts moving with constant speed at $t = 0$ from the piston; it is overtaken by a rarefaction wave starting at the time t_R when the piston is arrested. At the time $t = t_1$ of overtaking the interaction begins and the theory of Sec. 3 and 4 is to be applied from then on. The motion of the decaying shock is found to be given by

$$(15) \qquad t = t_R + (t_1 - t_R)\, k_1^2 \left(\frac{4-\sigma}{\sigma}\right)^2 , \quad \sigma_1 \geq \sigma > 0 ,$$

$$x = x_R + (1 + \sigma)\, c_o\, (t - t_R) ,$$

whence asymptotically, for large times t,

$$(16) \quad x \approx x_R + c_o[t - t_o + 4k_1 \sqrt{(t_1-t_R)(t-t_R)} - 4k_1^2(t_1-t_R)] .$$

(The values of the quantities t_1 and $k_1 = \sigma_1/(4-\sigma_1)$ are given in the Appendix.) The width of the shock wave, i.e. the distance between the shock front and the tail of the rarefaction wave is then immediately found to be asymptotically,

$$(17)\quad d(t) = c_o\,\sigma(t - t_R) = c_o \left\{4k_1 \sqrt{(t_1-t_R)(t-t_R)} - 4k_1^2(t_1-t_R)\right\}.$$

Thus it is seen that the process of interaction goes on indefinitely and that the width of the wave zone increases only like $\sqrt{t-t_R}$ and not like $t - t_R$ as the width of the undisturbed rarefaction wave.

The shock strength may be characterized by the excess pressure ratio which is asymptotically given by

$$(18)\quad \frac{p-p_o}{p_o} = (1 + \mu^2)\left\{\sigma + \frac{1}{2}\sigma^2 + \dots\right\}$$

$$\sim (1+\mu^2)\left\{4k_1\sqrt{\frac{t_1-t_R}{t-t_R}} + 4k_1^2\,\frac{t_1-t_R}{t-t_R}\right\}.$$

Thus we see that the shock strength decreases like const./$\sqrt{t-t_R}$.(*) The pressure distribution over the wave is asymptotically given by the quadratic distribution

$$(19)\quad \frac{p-p_o}{p_o} \sim (1 + \mu^2)\left\{\sigma + \frac{1}{2}\sigma^2\right\}$$

$$\sim (1 + \mu^2)\left\{\frac{x-x_R}{c_o(t-t_R)} - 1 + \frac{1}{2}\left(\frac{x-x_R}{c_o(t-t_R)} - 1\right)^2\right\},$$

(*) Chandrasekhar, who found this result in a special case, conjectured that this would be true in general.

for $0 \leq x - x_R - c_o(t - t_R) \leq d(t)$, (see Fig. 4). Eventually the distribution approaches a linear one.

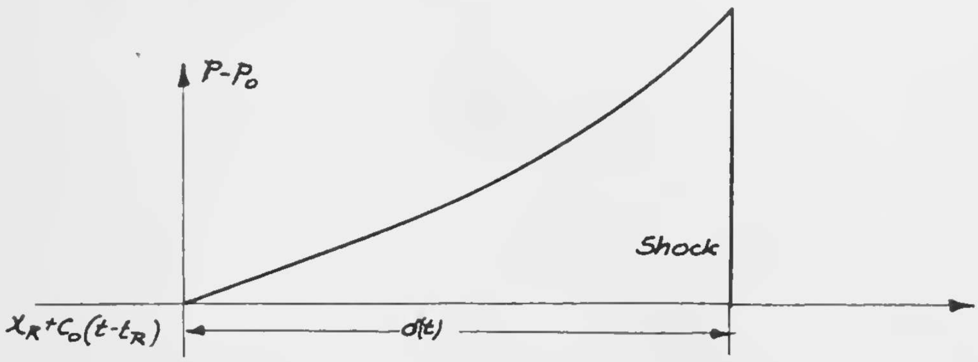

Fig. 4

Pressure distribution in decaying shock wave.

7. The method presented enables one quite generally to handle the problem of the flow resulting from any piston motion. Of particular interest is the case, in which <u>the piston is first pushed into the gas, then retracted, and finally arrested when it reaches the original position</u>; (cf. Appendix AIV). In this case the wave zone is eventually bordered by two shock fronts. For this reason such a wave motion has been termed "N-wave". Through the head shock, which was sent out when the piston was first set into motion, the pressure is raised; through the subsequent rarefaction wave the pressure drops to below the atmospheric value, to which it is raised again through the tail shock, which was sent out when the piston was finally arrested.

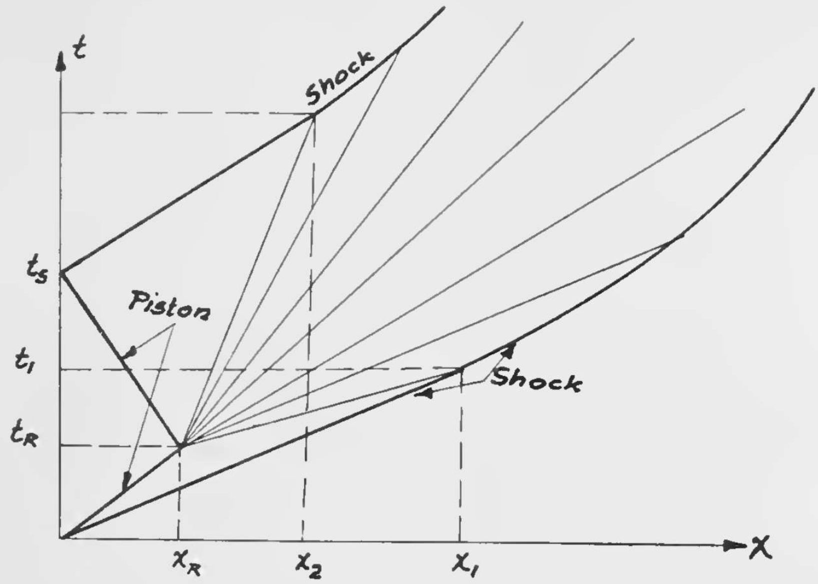

Fig. 5

Decaying N-wave with head and tail shock due to retraction of piston.

The width of the wave zone in this case is asymptotically given by

$$(20)\qquad d(t) = d_1(t) + d_2(t) ,$$

with

$$(21)\qquad \begin{cases} d_1(t) = 4c_o \left\{k_1 \sqrt{t_1-t_R}\sqrt{t-t_R} - k_1^2(t_1-t_R)\right\} \\ d_2(t) = 4c_o \left\{-k_2\sqrt{t_1-t_R}\sqrt{t-t_R} + k_2^2(t_1-t_R)\right\} \end{cases} ,$$

(note that $k_2 < 0$ and hence $d_2(t) > 0$, see Appendix IV); evidently $d(t)$ is growing like $\sqrt{t-t_R}$. The asymptotic pressure distribution is again given by (19), but extends over the wave zone

$$(22)\qquad -d_1(t) \leq x - x_R - c_o(t-t_R) \leq d_2(t) .$$

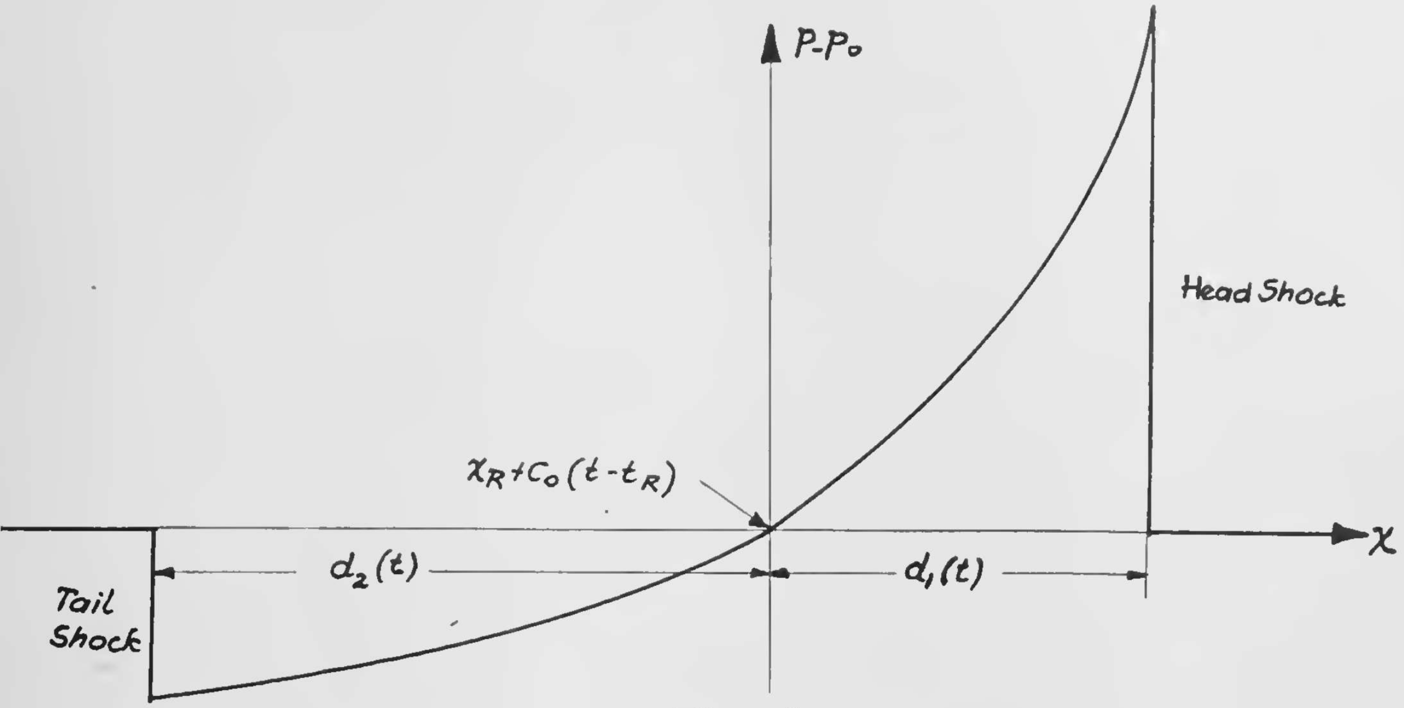

Fig. 6

Pressure distribution in decaying N-wave with head- and tail shock.

The notion of "N-wave" was introduced recently in a paper by DuMond, Cohen, Panofsky, and Deeds(*), who treated their propagation from a different approach. They derive formulas for the rates of change of the pressure in the wave and at the shock fronts and of the width of the wave, assuming that the wave is "balanced", i.e. that the product of the pressure ratios of head and tail shock equals one. (See (6), (7), (8) in their paper; in (6) and (7) a factor γ in the denominator was omitted.) These formulas are in agreement with the first order contributions to our formulas (19), (20). (21), from which they could be derived.

(*) See, "A Determination of the Wave Forms and Laws of Propagation and Dissipation of Ballistic Shock Waves" by J. W. M. DuMond, E. R. Cohen, W. K. H. Panofsky, and E. Deeds, reprinted from The Journal of the Acoustical Society of America, Vol. 18, No. 1, 97-118, July, 1946. See also L. D. Landau, J. Phys. Acad. Sci. U.S.S.R., Vol. 6, 229-230, 1942. In a paper by B. Cassen and J. Stanton, Theory of Non-Steady Shock Waves, of which an abstract appeared in the Bulletin of the American Physical Society, Vol. 22, No. 2, May 1947, similar problems are discussed by an approach apparently more closely related to ours.

8. One naturally asks how much faith may be put in the asymptotic representations obtained. A few critical remarks are called for. The method employed can be considered as giving the correct terms up to the second order of an expansion with respect to powers of the shock strength. Therefore the position of the shock front at a definite time t is given with arbitrarily good accuracy if the strength of the shock initially is made sufficiently small. One may be tempted to infer from this fact that our formulas for the position x of a shock front in a given problem represent the actual position with increasing accuracy for increasing time t, since the shock strength approaches zero as the time increases. This inference would, however, not be correct. For the case of the decaying shock wave treated in Sec. 6 this would mean, for example, that the coefficient k_1 in formula (16) was independent of the approximation method, which cannot be expected to be true. All that can be expected is that an asymptotic representation of the type (16) holds with coefficients not differing much from those in formula (16). Thus the deviation of the actual position x of the shock from the one given by (16) must be expected to increase like a quantity $\alpha\sqrt{t}$, but on the other hand the factor α may be expected to be small.

In order to determine the accuracy of the method one may refine the description of the shock wave by including terms of third order in the shock strength. Results of such an investigation will be reported in a later issue.

9. For the treatment of the problem of a shock front separating two simple waves (neither of which is a constant state) (cf. Fig. 3) the method explained in Sec. 4 is no longer applicable. According to the argument of Sec. 4, we may again assume that up to second order the simple waves are not influenced by the shock; but it does not seem possible to determine the motion of the shock front from a linear differential equation. Of particular interest is the case in which the shock begins with initial strength zero; such a situation arises, for example,

if a shock develops within a simple wave. In that case an expansion with respect to powers of the time will give sufficient information about the early stages of the development of the shock, (cf. Appendix AV).

Let us assume the simple wave to be given by one representation

$$(23) \qquad x = \omega t + A(\omega) ,$$

with

$$(24) \qquad \omega = u + c ;$$

u and c are then given by

$$(25) \qquad u = u_\nu + (1 - \mu^2)(\omega - \omega_\nu), \; c = c_\nu + \mu^2(\omega - \omega_\nu),$$

$$u_\nu + c_\nu = \omega_\nu$$

with appropriate constants u_ν and c_ν. Let us assume that the function $A(\omega)$ describing the initial distribution of ω satisfies the condition

$$(26) \qquad A'(\omega) < 0 ,$$

except for $\omega = \omega_\nu$ where

$$(27) \qquad A'(\omega_\nu) = A''(\omega_\nu) = 0, \; A'''(\omega_\nu) < 0 .$$

Then the straight forward characteristics form an envelope given through the parameter representation

$$(28) \qquad t = -A'(\omega), \; x = A(\omega) - \omega A'(\omega)$$

with a cusp at $t = 0$, $x = 0$. Consequently, a shock will develop. As will be shown in the Appendix AV, the expansion

of the representation of this shock begins with the terms

(29) $$x = \omega_\nu t + \frac{1}{5}(1 + k^{-1}\ell)c_\nu t^2/k \quad ,$$

in which

(30) $$k = -\frac{1}{6}\, c_\nu^2\, A'''(\omega_\nu),\ \ell = \frac{1}{24}\, c_\nu^3\, A''''(\omega_\nu) \ .$$

It is remarkable that up to the order considered the shock depends only on the initial distribution of $\omega = u + c$ and on the local value c_ν of c but not explicitly on the value of γ.

These formulas can be used to determine the formation of a shock in a compression wave which is produced when a piston is pushed into the gas beginning with acceleration zero. This will be explained in detail in Appendix AVI.

10. The decay of steady two-dimensional shock waves can be treated in much the same way as the decay of moving one-dimensional shock waves has been treated. The reason is that for steady two-dimensional waves, just as for one-dimensional unsteady waves, the transition relations for shocks and simple waves are the same up to terms of second order.

Consider a backward-facing simple wave that is forward inclined. Suppose further that the incoming supersonic flow with the constant velocity $u = q_o > c_o$, $v = 0$ is turned into a flow with velocity $u = q \cos \theta$, $v = q \sin \theta$, (see Fig. 7). The strength of this wave may be characterized by the angle θ and the changes of all quantities across the wave may be expanded in powers of θ. Similarly, consider a backward-facing forward-inclined shock front, (see Fig. 8) across which the flow with velocity $(q_o, 0)$ is turned into a flow with velocity $u = q \cos \theta$, $v = q \sin \theta$. Then again the changes of all quantities can be expanded in powers of θ. This expansion now agrees with the expansion for the simple wave up to terms of second order, (see Appendix BII). For this reason

we may in second approximation consider the flow behind the shock as if it had resulted through a simple wave.

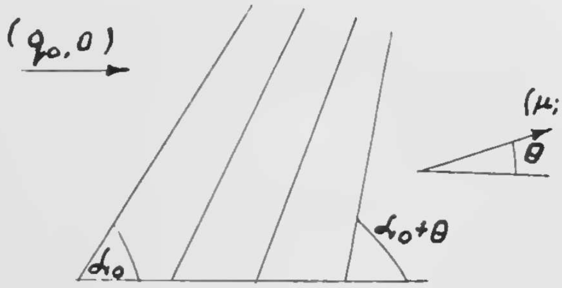

Fig. 7

Backward-facing forward-inclined simple wave indicated through a set of straight Mach lines.

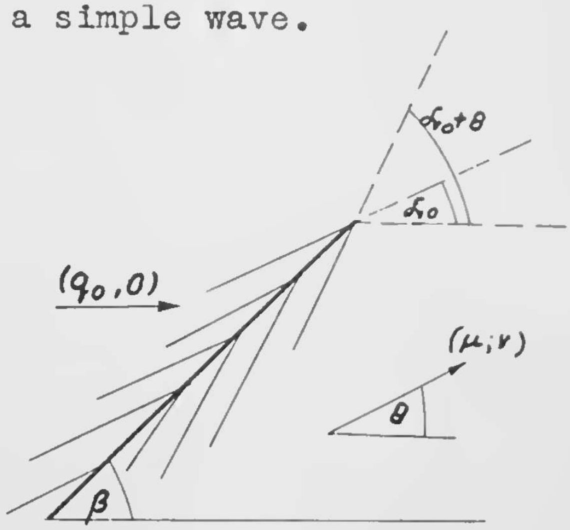

Fig. 8

Backward-facing forward-inclined shock. Forward-inclined Mach lines are shown on both sides.

This fact may also be expressed as follows: consider in the (u,v)-plane the shock polar and the epicycloid through the same point $(q_o, 0)$ employing the same value for the critical speed c_*, (see Manual III, 35, and IV, 61, 69). Then these two curves agree up to second order in the sense that their representations in terms of the parameter θ agree up to second order. If, therefore, the flow velocity u,v behind a shock with a certain angle θ of flow direction, represented by a point on the shock polar, is replaced by a velocity with the same angle θ represented by a point on the epicycloid, then the shock relations for u and v are satisfied up to terms of second order in the shock strength. The sound speed c is determined from Bernoulli's law $\mu^2 q^2 + (1 - \mu^2)c^2 = c_*^2$ and p and ρ are functions of c, if we assume the entropy to be the same on both sides.

Suppose now <u>a supersonic flow along a straight wall with</u>

a bump is to be determined, (see Fig. 9). We may, for example, consider the upper part of an airfoil cross section as such a bump. We assume that the incoming flow is super-

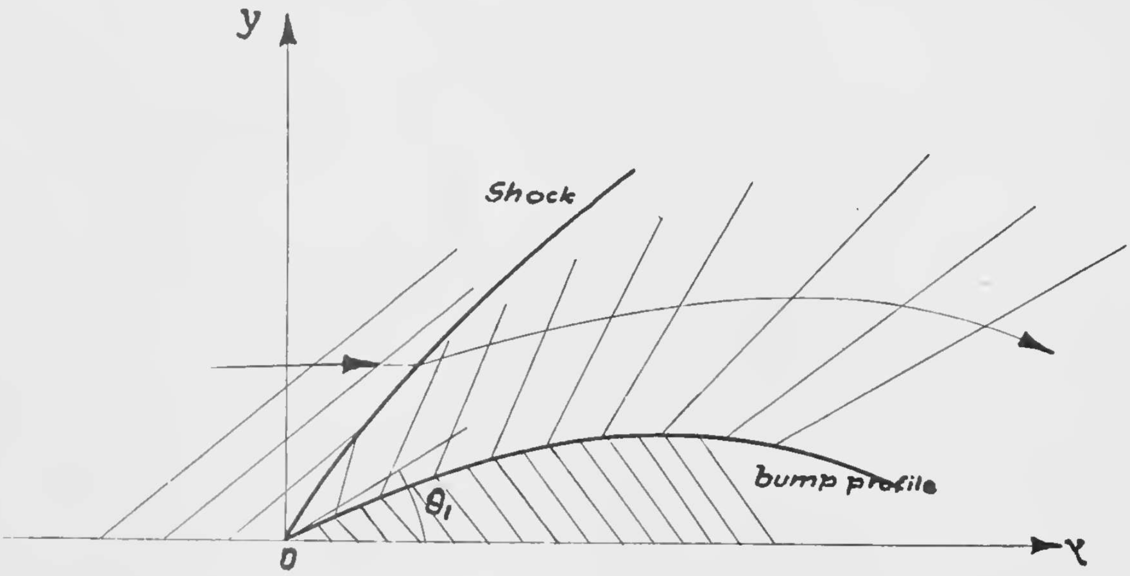

Fig. 9

Flow along a straight wall with a bump. The resulting shock and forward-facing Mach lines are shown.

sonic $M_o = q_o/c_o > 1$, that the profile of the bump begins with a sharp angle θ_1 of inclination against the straight wall, which we take as x-axis, and that θ_1 is so small that the turning of the flow through the angle θ_1 may be achieved by a shock starting at the tip. The continuation of this shock varies in strength and inclination (unless the bump is straight); hence the flow behind the shock front is rotational and carries a non-constant entropy. In second order, however, this flow is just the simple wave flow that would have resulted if the flow had turned through the angle θ_1 by means of a simple compression wave. This simple wave flow corresponds in the (u,v)-plane to the epicycloid which begins at the point $(q_o,0)$ and carries the same entropy as the flow ahead of the shock. Once this simple wave has been determined we shall obtain approximately the shock line as the solution

of an appropriate linear differential equation. This equation is based on an expression, valid up to terms of second order, for the slope of the shock line in terms of the states on both sides. This shock line cuts off a part of the full simple wave flow. The flow so described satisfies all conditions correctly up to the second order since the states on both sides of the shock line are represented in the (u,v)-plane by points on the same epicycloid and have the same entropy.

In order to determine the simple wave flow that represents the flow behind the shock line in second approximation, we suppose that the bump profile is given by a representation

(31) $$y = Y(x), \ x \geq 0; \quad Y(0) = 0 \ .$$

Through every point $x = \xi$ of the profile a straight Mach line passes, given by the equation

(32) $$y = t(x - \xi) + Y(\xi)$$

in which the slope t of the Mach line is a given function $t = t(\theta)$ of the angle of inclination θ of the flow with the x-axis. Since the gas moves along the profile, this angle θ is connected with the function $Y(\xi)$ through

(33) $$\tan \theta = Y'(\xi) \ .$$

The slope t is given by the relation

(34) $$t = \tan(\alpha + \theta) \ ,$$

in which α is the Mach angle, connected with the Mach number $M = q/c$ through

(35) $$M = 1/\sin\alpha \quad \text{or} \quad 1/\sqrt{M^2-1} = \tan\alpha \ .$$

The Mach number M in turn can be expressed through the flow speed q and the connection between the flow speed q and the angle θ of flow direction can be read off from the epicycloid through the point (q_o, v). In this way t can be determined as function of θ.

To determine the shock line we may employ any relation which expresses the slope

$$s = \tan \beta \tag{36}$$

of the shock, or its angle of inclination β, in terms of quantities on both sides of it. It is natural to use a relation connecting s with the angle θ. Along the shock line we have

$$\frac{dy}{dx} = s \quad . \tag{37}$$

Inserting s as function of θ, and θ as function of ξ, into the relation

$$\frac{dy}{dx}\frac{dx}{d\xi} = t\left(\frac{dx}{d\xi} - 1\right) + \frac{dt}{d\xi}(x - \xi) + Y'(\xi) \; ,$$

derived from (32), we obtain the linear differential equation

$$(t - s)\frac{dx}{d\xi} + \frac{dt}{d\xi}\, x = t + \xi \frac{dt}{d\xi} - Y'(\xi) \; , \tag{38}$$

for x as function of ξ, which is to be solved under the initial condition

$$x = 0 \quad \text{for} \quad \xi = 0 \; . \tag{39}$$

For the solution of this differential equation it is convenient to expand t in powers of the angle θ and to retain only the terms up to the second order. We then obtain, (see Appendix BIII)

$$t = t_o + a_1\theta + a_2 t_o^{-1}\theta^2 + \ldots \tag{40}$$

with

$$a_1 = \frac{1}{1-\mu^2}(1 + t_o^2)^2 \tag{41}$$

$$a_2 = \frac{1}{(1-\mu^2)^2}\, t_o^2(1 + t_o^2)^2(1 + \mu^2 + 2t_o^2) .$$

We note that t_o is connected with the Mach number M_o of the incoming stream through

$$M_o^{-2} = t_o^2(1 + t_o^2)^{-1} \quad \text{or} \quad t_o^2 = M_o^{-2}(1 - M_o^{-2})^{-1} . \tag{42}$$

We further expand s in powers of $t - t_o$, obtaining

$$s = t_o + \tfrac{1}{2}(t - t_o) + \tfrac{1}{2}k\, t_o^{-1}(t - t_o)^2 \tag{43}$$

with

$$k = \frac{1}{4}\,\frac{1-3t_o^2}{1+t_o^2} = \frac{1}{4} - M_o^{-2} . \tag{44}$$

Equation (38) then takes the form

$$\frac{dx}{dt} + \frac{2x}{(t-t_o)-kt_o^{-1}(t-t_o)^2} \tag{45}$$

$$= \frac{2}{(t-t_o)-kt_o^{-1}(t-t_o)^2}\,\frac{d}{dt}\,(\xi t - Y(\xi)) .$$

The solution of this equation is

(46)
$$x(\xi) = 2\left(\frac{1-kt_o^{-1}(t-t_o)}{t-t_o}\right)^2 \cdot \int_o^{\xi} \frac{t'-t_o}{[1-kt_o^{-1}(t'-t_o)]^3}\, d[\xi' t' - Y(\xi')]$$

in which $t = t(\textcircled{H}(\xi))$ and $t' = t(\textcircled{H}(\xi'))$; $y(\xi)$ is then obtained by insertion in (32).

A general conclusion can be drawn from this formula. When the parameter ξ approaches a value $\xi = x_m$ for which Θ approaches zero and hence t approaches t_o the factor in front of the integral becomes ∞, and thus both x and y become infinite. In other words the shock front penetrates only into that part of the simple wave which issues from the section of the bump between its beginning $\xi = 0$, and its peak, $\xi = x_m$. The shock line extends to infinity without ever reaching the Mach line which issues from this peak and asymptotically it becomes parallel to this Mach line.

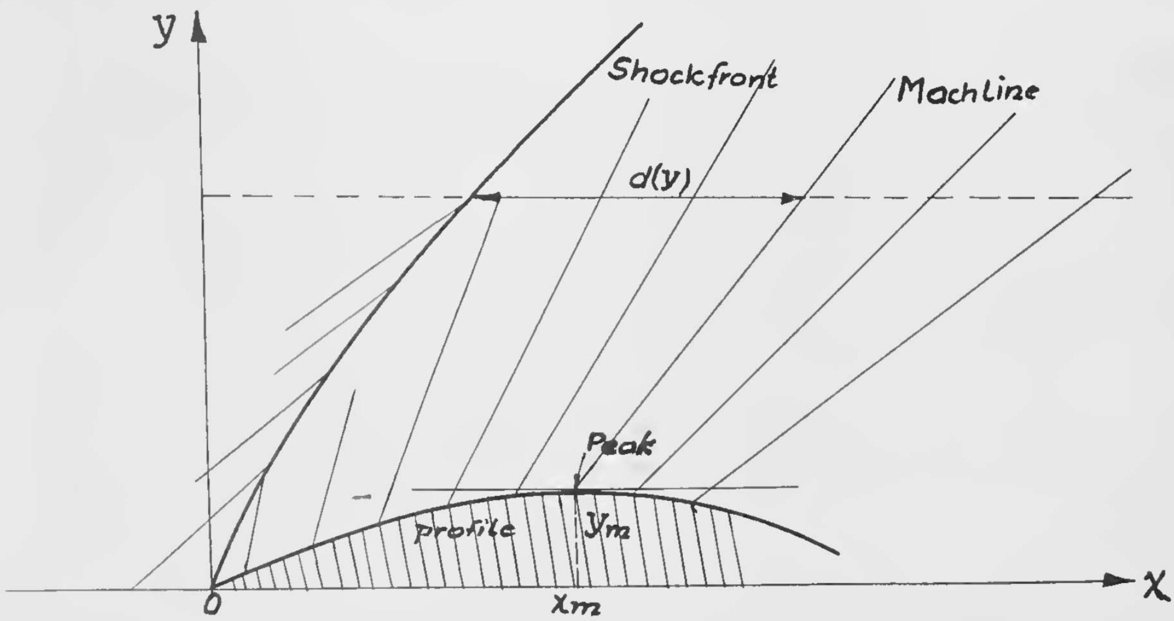

Fig. 10

The shock front will asymptotically become parallel to the Mach line issuing from the peak.

To investigate more closely the asymptotic behavior of the shock front at great distances from the axis we expand all quantities with respect to negative powers of y. We introduce the quantity δ (*) by

(47) $$\delta = 2t_o^{-2} \int_o^{x_m} (t-t_o)[1-kt_o^{-1}(t-t_o)]^{-3} d(\xi t - Y(\xi))$$

and find from (46) .

(48) $$x = \delta \left\{ t_o^2 (t-t_c)^{-2} - 2kt_o(t-t_o)^{-1} + \dots \right. ,$$

from which

(49) $$t-t_o = t_o \left\{ \delta^{1/2} x^{-1/2} - k\delta x^{-1} + \dots \right. .$$

Inserting relation (49) in (32) gives

(50) $$y = y_m + t_o(x-x_m) + t_o \delta^{1/2} x^{1/2} - t_o k \delta + \dots$$

Expressing x in terms of y we find

(51) $$x = x_m + t_o^{-1}(y-y_m) - d(y) ,$$

in which

(52) $$d(y) = \delta^{1/2} t_o^{-1/2} y^{1/2} - (\tfrac{1}{2} + k)\delta + \dots .$$

(*) If the profile is given by

$$y = y_m \left(2 \frac{x}{x_m} - \left(\frac{x}{x_m}\right)^2\right)$$

we have,

$$\delta = 2a_1 t_o^{-1} y_m + \tfrac{4}{3}(a_1^2 - 2a_1 + 6ka_1^2 + 2a_2) t_o^{-2} y_m^2 + \dots .$$

This quantity $d = d(y)$ is the distance the shock front stands ahead of the Mach line $x = x_m + t_o^{-1}(y-y_m)$ issuing from the top of the profile; we see that this distance increases like the square root of the distance y from the axis.

The strength of the shock is in first order proportional to $t-t_o$; hence the strength of the shock decreases like the reciprocal of the square root of the distance from the y-axis. To calculate this strength as a function of y one may proceed as follows. One inserts (51) into (49), and (49) into (43); then one uses the formula

$$(53) \qquad \frac{p-p_o}{\rho\, q_o^2} = \frac{\mu^2+s^2}{1+s^2} - \frac{\mu^2+t_o^2}{1+t_o^2} \quad ,$$

(cf. Manual (52) IV-69, p. 193). The expansion of the relative pressure increase $(p-p_o)/p_o$ with respect to negative powers of $y^{1/2}$ could easily be derived. The term of first order obtained this way is

$$(54) \qquad \frac{p-p_o}{p_o} = (1+\mu^2)\, \frac{t_o^{1/2}\,\delta^{1/2}}{1+t_o^2}\, y^{-1/2} + \dots$$

Also the gas velocity behind the shock and the flow angle θ are easily expressed in terms of $s = \tan\beta$, (cf. (50), (52), IV-69, p. 192, 193). The angle θ as a function of y could be more directly obtained by inverting relation (40).

The critical remarks made in Sec. 8 apply in an obvious manner to the problem treated in this article. Dumond, Cohen, Panofsky and Deeds (see p. 13) have made the very interesting remark that the width of the shock wave is determined asymptotically solely by the shock strength and the distance from the profile, independently of the specific shape of the profile. We find the same result from (52) and (54) in first order:

$$\frac{p-p_o}{p_o} = (1 + \mu^2)\,\frac{t_o}{1+t_o^2}\,\frac{d(y)}{y}\,. \tag{55}$$

The coefficient here differs, however, from the coefficient given in that paper by the factor $2t_o/\sqrt{1+t_o^2} = 2\sin\alpha$ (see l.c. (9)).